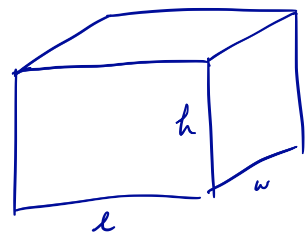
Example 3. A rectangular box is to be made from 100 m^2 of cardboard. Find the maximum volume of such a box.



We want to maximize $V = lwh$

where $2lw + 2lh + 2wh = 100$ ①

and $l, w, h > 0$

Solve for h in ①: $2lh + 2wh = 100 - 2lw$

$$\Rightarrow h = \frac{100 - 2lw}{2l + 2w} = \frac{50 - lw}{l + w}$$

$$\Rightarrow \text{We want to maximize } V = \frac{lw(50 - lw)}{l + w} = \frac{50lw - l^2w^2}{l+w} \leftarrow \text{define } f(l,w)$$

$$f_l(l,w) = \frac{(l+w)(50w - 2lw^2) - (50lw - l^2w^2)(1)}{(l+w)^2}$$

$$= \frac{\cancel{50lw} + 50w^2 - 2l^2w^2 - 2lw^3 - \cancel{50lw} + l^2w^2}{(l+w)^2}$$

$$= \frac{50w^2 - l^2w^2 - 2lw^3}{(l+w)^2} = \frac{w^2(50 - l^2 - 2lw)}{(l+w)^2}$$

In a similar fashion, $f_w(l,w) = \dfrac{l^2(50 - w^2 - 2lw)}{(l+w)^2}$

Find critical points:

$$\left.\begin{aligned} -w^2(\ell^2+2\ell w-50) &= 0 \quad (1) \\ -\ell^2(w^2+2\ell w-50) &= 0 \quad (2) \end{aligned}\right\}$$

$(1) \Rightarrow w = 0 \; (3)$ or $\ell^2+2\ell w-50=0 \; (4)$

$(2) \Rightarrow \ell = 0 \; (5)$ or $w^2+2\ell w-50=0 \; (6)$

$(3)+(5) \Rightarrow w=0, \ \ell=0$

$(3)+(6) \Rightarrow 0+2\ell(0)-50=0 \Rightarrow$ impossible

$(4)+(5) \Rightarrow 0+2w(0)-50=0 \Rightarrow$ impossible

$(4)+(6) \Rightarrow$ Solve for $2\ell w$

$\Rightarrow \ell^2-50 = w^2-50 \Rightarrow \ell = w$

Sub into $(4) \Rightarrow 3\ell^2=50 \Rightarrow \ell=\sqrt{\frac{50}{3}}$

$\Rightarrow w=\sqrt{\frac{50}{3}}, \ \ell=\sqrt{\frac{50}{3}}$

$\Rightarrow$ critical points: $\left(\sqrt{\frac{50}{3}}, \sqrt{\frac{50}{3}}\right)$. (since $\ell, w > 0$)

Second derivatives test? Instead...

By the physical nature of the problem, there must be an absolute maximum, which must also be a local maximum, and so must occur at a critical point!

$$\Rightarrow f\left(\sqrt{\tfrac{50}{3}}, \sqrt{\tfrac{50}{3}}\right) = \frac{\frac{50}{3}\left(50-\frac{50}{3}\right)}{2\sqrt{\frac{50}{3}}} = \frac{\frac{50}{3}\cdot\frac{100}{3}}{2\sqrt{\frac{50}{3}}} = \frac{50}{3}\sqrt{\frac{50}{3}}$$ is an absolute maximum

$\Rightarrow$ The max. volume of a box made of $100\,cm^3$ of cardboard is $\frac{50}{3}\sqrt{\frac{50}{3}}\ cm^3$.